MATRICES

An arrangement of numbers in rows and columns. A matrix of type "($m \times n$)" is defined as arrangement of ($m \times n$) numbers in 'm' rows & 'n' columns. Usually these numbers are enclosed within square brackets [ ] (or) simple brackets ( ) are denoted by capital letters A, B, C etc.

Example

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 9 & 10 & -1 & 3 \\ 4 & 2 & 8 & 5 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 3 & 4 \\ 2 & 10 & 2 \\ 9 & -1 & 3 \\ 4 & 8 & 5 \end{pmatrix}$$

Here A is of type 3 x 4 & B is of type 4 x 3

Types of matrices

1. Row matrix: It is a matrix containing only one row and several columns .It is also called as row vector.

Example:

$$\begin{bmatrix} 1 & 3 & 7 & 9 & 6 \end{bmatrix}$$

↓

(1 x 5) matrix called row vector.

2. Column matrix: It is a matrix containing only one column. It is also known as column vector.

Example: $\begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix}$ (3x1)

3. Square matrix: A matrix is called as square matrix, if the number of rows is equal to number of columns.

$$\begin{pmatrix} 4 & 2 & 4 \\ 1 & 9 & 8 \\ 6 & 5 & 2 \end{pmatrix}$$

Example

The elements a_{11}, a_{22}, a_{33} etc fall along the diagonal & this is called a leading diagonal (or) principal diagonal of the matrix.

4. Trace of the matrix

It is defined as the sum of the elements along the leading diagonal.

In this above matrix the trace of the matrix is

4 + 9 + 2 = 15.

5. Diagonal matrix

It is a square matrix in which all the elements other than in the leading diagonals are zero's.

Eg: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

6.Scalar matrix

It is a diagonal matrix in which all the elements in the leading diagonal are same.

Eg: $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

7. Unit matrix or identify matrix

It is a diagonal matrix, in which the elements along the leading diagonal are equal to one. It is denoted by I

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

8. Zero matrix (or) Non-matrix

It is matrix all of whole elements are equal to zero denoted by "O"

Eg: $O = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ 2 x 3 $O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ 2 x 2

9. Triangular matrix

There are two types. 1. Lower Triangular Matrix 2. Upper Triangular Matrix.

Lower Triangular matrix

It is a square matrix in which all the elements above the leading diagonal are zeros.

Eg: $\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 4 & 6 & 2 \end{pmatrix}$

Upper Triangular matrix

Square matrix in which all the elements below the leading diagonal are zeros

Eg: $\begin{pmatrix} 5 & 2 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & 4 \end{pmatrix}$

10. Symmetric matrix

A square r = $\{a_{ij}\}$ said i = 1 to n ; j = 1 to n said to symmetric, if $a_{ij} = a_{ij}$ for all i and j.

Eg: $\begin{pmatrix} 1 & 3 & 4 \\ 3 & 6 & -5 \\ 4 & -5 & 2 \end{pmatrix}$

11. Skew symmetric matrix

A square matrix A = $\{a_{ij}\}$ i = 1 . to n is called skew symmetric, if $a_{ij} = -a_{ji}$ for all i & j. Here $a_{ii} = 0$ for all i

Eg: $\begin{pmatrix} 0 & 3 & -4 \\ -3 & 0 & 5 \\ 4 & -5 & 0 \end{pmatrix}$

Algebra of matrices

1. Equality of matrices

Two matrices A & B are equal, if and only if,

(i) Both A & B are of the same type

(ii) Every element of 'B' is the same as the corresponding element of 'A'.

Example

1.

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 9 & 10 & -1 & 3 \\ 4 & 2 & 8 & 5 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 3 & 4 \\ 2 & 10 & 2 \\ 9 & -1 & 3 \\ 4 & 8 & 5 \end{pmatrix}$$

Here order of matrix A is not same as order matrix B, the two matrices are not equal.

$A \neq B$

2. Find the value of *a* and *b* given

$$\begin{bmatrix} 4 & 5 \\ a & b \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ 3 & 2 \end{bmatrix}$$

Solution:

The given matrices are equal

$\therefore$ $a = 3$, $b = 2$

2. Addition of matrices

Two matrices A & B can be added if and only if,

(i) Both are of the same type.

(ii) The resulting matrix of A & B is also of same type and is obtained by adding the all elements of 'A' to the corresponding elements of 'B'.

Example

1. Find $\begin{bmatrix} 4 & 5 \\ 5 & 6 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}$

Solution

$$\begin{bmatrix} 4 & 5 \\ 5 & 6 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 4+2 & 5+3 \\ 5+2 & 6+1 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 7 & 7 \end{bmatrix}$$

3. Subtraction of the matrices

This can be done, when both the matrices are of same type.

(A-B) is obtained by subtracting the elements of 'A' with corresponding elements of 'B'.

Example

1. Find $\begin{bmatrix} 4 & 5 \\ 5 & 6 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}$

Solution

$$\begin{bmatrix} 4 & 5 \\ 5 & 6 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 4-2 & 5-3 \\ 5-2 & 6-1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 3 & 5 \end{bmatrix}$$

4. Multiplication of matrix

They are of two types : 1. By a scalar K B.

2. By a matrix $\longrightarrow$ A x B.

i) Scalar multiplication

To multiply a matrix 'A' by a scalar 'K', then multiply every element of a matrix 'A' by that scalar.

Example

1. Find $2\begin{bmatrix} 4 & 5 \\ 5 & 6 \end{bmatrix}$

Solution:

$$2\begin{bmatrix} 4 & 5 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 8 & 10 \\ 10 & 12 \end{bmatrix}$$

ii) Matrix Multiplication

Two matrices A & B can be multiplied to form the matrix product AB, if and only if the number of columns of 1st matrix A is equal to the number of rows of 2nd matrix B. If A is an *(m x p)* and B is an *(p x n)* then the matrix product AB can be formed. AB is a matrix by *(m x n)*.

In this case the matrices A and B are said to be conformable for matrix multiplication.

Example

1. Find $\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} 6 & 4 \\ 2 & -3 \end{pmatrix}$

Solution

$$\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} 6 & 4 \\ 2 & -3 \end{pmatrix} = \begin{bmatrix} 2\times 6+3\times 2 & 2\times 4+3\times -3 \\ 4\times 6+5\times 2 & 4\times 4+5\times -3 \end{bmatrix}$$

$$= \begin{bmatrix} 12+6 & 8+-9 \\ 24+10 & 16+-15 \end{bmatrix}$$

$$= \begin{bmatrix} 18 & -1 \\ 34 & 1 \end{bmatrix}$$

Note: The matrix product AB is different from the matrix product BA.

1. The matrix AB can be formed but not BA

   Eg: A is a (2 x 3) matrix

   B is a (3 x 5) matrix

   AB alone can be formed and it is a (2 x 5) matrix.

2. Even if AB & BA can be formed, they need not be of same type.

   Eg: A is a (2 x 3) matrix

   B is a (3 x 2) matrix

   AB can be formed and is a (2 x 2) matrix

   BA can be formed and is a (3x 3) matrix

3. Even if AB & BA are of the same type, they needn't be equal. Because, they need not be identical.

   Eg: A is a (3 x 3) matrix

   B is a (3 x 3) matrix

   AB is a (3 x 3) matrix

   BA is a (3x 3) matrix

   $AB \neq BA$

The multiplication of any matrix with null matrix the resultant matrix is also a null matrix.

When any matrix (ie.) A is multiplied by unit matrix; the resultant matrix is 'A' itself.

Transpose of a matrix

The Transpose of any matrix ('A') is obtained by interchanging the rows & columns of 'A' and is denoted by A^T. If A is of type *(m x n),* then A^T is of type *(n x m).*

Eg: $A = \begin{pmatrix} 2 & 3 \\ 1 & 0 \\ 4 & 5 \end{pmatrix}$ (3 x 2) $\quad A^T = \begin{pmatrix} 2 & 1 & 4 \\ 3 & 0 & 5 \end{pmatrix}$ (2 x 3)

Properties of transpose of a matrix

1) $(A^T)^T = A$

2) $(AB)^T = B^T A^T$ is known as the reversal Law of Transpose of product of two matrices.

DETERMINANTS

Every square matrix A of order n x n with entries real or complex there exists a number called the determinant of the matrix A denoted by by $|A|$ or det (A). The determinant formed by the elements of A is said to be the determinant of the matrix A..

Consider the 2nd order determinant.

$$|A| = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

Eg:

$$|A| = \begin{vmatrix} 4 & 3 \\ 1 & 0 \end{vmatrix} = 0\text{-}3 = \text{-}3$$

Consider the 3rd order determinant,

$$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

This can be expanded along any row or any column. Usually we expand by the 1st row. On expanding along the 1st row

$$|A| = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

Minors

Let A = (a_{ij})be a determinant of order n. The minor of the element a_{ij} is the determinant formed by deleting ith row and jth column in which the element belongs and the cofactor of the element is $A_{ij} = (-i)^{i+j} M_{ij}$ where M is the minor of ith row and jth column .

Example 1 Calculate the determinant of the following matrices.

(*a*) $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 4 \\ 1 & 3 & -1 \end{pmatrix}$

Solution

$$\begin{vmatrix} 1 & 2 & 3 \\ 3 & 2 & 4 \\ 1 & 3 & -1 \end{vmatrix} = 1\begin{vmatrix} 2 & 4 \\ 3 & -1 \end{vmatrix} - 2\begin{vmatrix} 3 & 4 \\ 1 & -1 \end{vmatrix} + 3\begin{vmatrix} 3 & 2 \\ 1 & 3 \end{vmatrix}$$

$$= 1(-2-12) - 2(-3-4) + 3(9-2)$$
$$= -14 + 14 + 21 = 21$$

Singular and Non-Singular Matrices:

Definition

A square matrix 'A' is said to be singular if, $|A| = 0$ and it is called non-singular if $|A| \neq 0$.

Note

Only square matrices have determinants.

Example: Find the solution for the matrix $A = \begin{vmatrix} 2 & 4 & 3 \\ 5 & 1 & 0 \\ 7 & 5 & 3 \end{vmatrix}$

$$\begin{vmatrix} 2 & 4 & 3 \\ 5 & 1 & 0 \\ 7 & 5 & 3 \end{vmatrix} = 2\begin{vmatrix} 1 & 0 \\ 5 & 3 \end{vmatrix} - 4\begin{vmatrix} 5 & 0 \\ 7 & 3 \end{vmatrix} + 3\begin{vmatrix} 5 & 1 \\ 7 & 5 \end{vmatrix}$$

$$= 2(3-0) - 4(15-0) + 3(25-7)$$
$$= 6 - 60 + 54 = 0$$

Here $|A| = 0$ .So the given matrix is singular

Properties of determinants

1. **The value of a determinant is unaltered by interchanging its rows and columns.**

Example

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 4 \\ 1 & 3 & -1 \end{bmatrix}$ then

$$\det(A) = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 2 & 4 \\ 1 & 3 & -1 \end{vmatrix} = 1\begin{vmatrix} 2 & 4 \\ 3 & -1 \end{vmatrix} - 2\begin{vmatrix} 3 & 4 \\ 1 & -1 \end{vmatrix} + 3\begin{vmatrix} 3 & 2 \\ 1 & 3 \end{vmatrix}$$

$$= 1(-2-12) - 2(-3-4) + 3(9-2)$$
$$= -14 + 14 + 21 = 21$$

Let us interchange the rows and columns of A. Thus we get new matrix $A_{1.}$

Then

$$\det(A_1) = \begin{vmatrix} 1 & 3 & 1 \\ 2 & 2 & 3 \\ 3 & 4 & -1 \end{vmatrix} = 1\begin{vmatrix} 2 & 3 \\ 4 & -1 \end{vmatrix} - 3\begin{vmatrix} 2 & 3 \\ 3 & -1 \end{vmatrix} + 1\begin{vmatrix} 2 & 2 \\ 3 & 4 \end{vmatrix}$$

$$= 1(-2-12) - 3(-2-9) + 1(8-6)$$

$$= -14 + 33 + 2 = 21$$

Hence det (A) = det (A_1).

2. If any two rows (columns) of a determinant are interchanged the determinant changes its sign but its numerical value is unaltered.

Example

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 4 \\ 1 & 3 & -1 \end{bmatrix}$ then

$$\det(A) = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 2 & 4 \\ 1 & 3 & -1 \end{vmatrix} = 1\begin{vmatrix} 2 & 4 \\ 3 & -1 \end{vmatrix} - 2\begin{vmatrix} 3 & 4 \\ 1 & -1 \end{vmatrix} + 3\begin{vmatrix} 3 & 2 \\ 1 & 3 \end{vmatrix}$$

$$= 1(-2-12) - 2(-3-4) + 3(9-2)$$

$$= -14 + 14 + 21 = 21$$

Let A_1 be the matrix obtained from A by interchanging the first and second row. i.e R1 and R2.

Then

$$\det(A_1) = \begin{vmatrix} 3 & 2 & 4 \\ 1 & 2 & 3 \\ 1 & 3 & -1 \end{vmatrix} = 3\begin{vmatrix} 2 & 3 \\ 3 & -1 \end{vmatrix} - 2\begin{vmatrix} 1 & 3 \\ 1 & -1 \end{vmatrix} + 4\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix}$$

$$= 3(-2-9) - 2(-1-3) + 4(3-2)$$

$$= -33 + 8 + 4 = -21$$

Hence det (A) = - det (A_1).

3. If two rows (columns) of a determinant are identical then the value of the terminant is zero.

Example

Let $A = \begin{bmatrix} 1 & 1 & 3 \\ 3 & 3 & 4 \\ 1 & 1 & -1 \end{bmatrix}$ then

$$\det(A)=\begin{vmatrix}1 & 1 & 3\\ 3 & 3 & 4\\ 1 & 1 & -1\end{vmatrix}=1\begin{vmatrix}3 & 4\\ 1 & -1\end{vmatrix}-1\begin{vmatrix}3 & 4\\ 1 & -1\end{vmatrix}+3\begin{vmatrix}3 & 3\\ 1 & 1\end{vmatrix}$$

$$=1(-3-4)-1(-3-4)+3(3-3)$$

$$=-7+7+0=0$$

Hence $\det(A)=0$

4. If every element in a row (or column) of a determinant is multiplied by a constant "k" then the value of the determinant is multiplied by k.

Example

Let $A=\begin{bmatrix}1 & 2 & 3\\ 3 & 2 & 4\\ 1 & 3 & -1\end{bmatrix}$ then

$$\det(A)=\begin{vmatrix}1 & 2 & 3\\ 3 & 2 & 4\\ 1 & 3 & -1\end{vmatrix}=1\begin{vmatrix}2 & 4\\ 3 & -1\end{vmatrix}-2\begin{vmatrix}3 & 4\\ 1 & -1\end{vmatrix}+3\begin{vmatrix}3 & 2\\ 1 & 3\end{vmatrix}$$

$$=1(-2-12)-2(-3-4)+3(9-2)$$

$$=-14+14+21=21$$

Let A_1 be the matrix obtained by multiplying the elements of the first row by 2 (ie. here k =2) then

$$\det(A_1)=\begin{vmatrix}2(1) & 2(2) & 2(3)\\ 3 & 2 & 4\\ 1 & 3 & -1\end{vmatrix}=2\times1\begin{vmatrix}2 & 4\\ 3 & -1\end{vmatrix}-2\times2\begin{vmatrix}3 & 4\\ 1 & -1\end{vmatrix}+2\times3\begin{vmatrix}3 & 2\\ 1 & 3\end{vmatrix}$$

$$=2[1(-2-12)-2(-3-4)+3(9-2)]$$

$$=2[-14+14+21]=2(21)$$

Hence det (A) = 2 det (A_1).

5. If every element in any row (column) can be expressed as the sum of two quantities then given determinant can be expressed as the sum of two determinants of the same order with the elements of the remaining rows (columns) of both being the same.

Example

Let $A=\begin{bmatrix}1+2 & 2+4 & 3+6\\ 3 & 2 & 4\\ 1 & 3 & -1\end{bmatrix}$ then

$$\det(A) = \begin{vmatrix} 1+2 & 2+4 & 3+6 \\ 3 & 2 & 4 \\ 1 & 3 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 2 & 4 \\ 1 & 3 & -1 \end{vmatrix} + \begin{vmatrix} 2 & 4 & 6 \\ 3 & 2 & 4 \\ 1 & 3 & -1 \end{vmatrix}$$

$$= \det(M1) + \det(M2)$$

$$\det(M1) = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 2 & 4 \\ 1 & 3 & -1 \end{vmatrix} = 1\begin{vmatrix} 2 & 4 \\ 3 & -1 \end{vmatrix} - 2\begin{vmatrix} 3 & 4 \\ 1 & -1 \end{vmatrix} + 3\begin{vmatrix} 3 & 2 \\ 1 & 3 \end{vmatrix}$$

$$= 1(-2-12) - 2(-3-4) + 3(9-2)$$

$$= -14 + 14 + 21 = 21$$

$$\det(M2) = \begin{vmatrix} 2 & 4 & 6 \\ 3 & 2 & 4 \\ 1 & 3 & -1 \end{vmatrix} = 2\begin{vmatrix} 2 & 4 \\ 3 & -1 \end{vmatrix} - 4\begin{vmatrix} 3 & 4 \\ 1 & -1 \end{vmatrix} + 6\begin{vmatrix} 3 & 2 \\ 1 & 3 \end{vmatrix}$$

$$= 2(-2-12) - 4(-3-4) + 6(9-2)$$

$$= -28 + 28 + 42 = 42$$

$$\det(A) = \det(M1) + \det(M2)$$

$$\textit{Hence} \ \det(A) = 21 + 42 = 63$$

6. A determinant is unaltered when to each element of any row (column) is added to those of several other rows (columns) multiplied respectively by constant factors.

Example

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 4 \\ 1 & 3 & -1 \end{bmatrix}$ then

$$\det(A) = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 2 & 4 \\ 1 & 3 & -1 \end{vmatrix} = 1\begin{vmatrix} 2 & 4 \\ 3 & -1 \end{vmatrix} - 2\begin{vmatrix} 3 & 4 \\ 1 & -1 \end{vmatrix} + 3\begin{vmatrix} 3 & 2 \\ 1 & 3 \end{vmatrix}$$

$$= 1(-2-12) - 2(-3-4) + 3(9-2)$$

$$= -14 + 14 + 21 = 21$$

Let A_1 be a matrix obtained when the elements C_1 of A are added to those of second column and third column multiplied respectively by constants 2 and 3. Then

$$\det(A_1) = \begin{vmatrix} 1+2(2)+3(3) & 2 & 3 \\ 3+2(2)+3(4) & 2 & 4 \\ 1+2(3)+3(-1) & 3 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 2 & 4 \\ 1 & 3 & -1 \end{vmatrix} + \begin{vmatrix} 2(2) & 2 & 3 \\ 2(2) & 2 & 4 \\ 2(3) & 3 & -1 \end{vmatrix} + \begin{vmatrix} 3(3) & 2 & 3 \\ 3(4) & 2 & 4 \\ 3(-1) & 3 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 2 & 3 \\ 3 & 2 & 4 \\ 1 & 3 & -1 \end{vmatrix} + 2\begin{vmatrix} 2 & 2 & 3 \\ 2 & 2 & 4 \\ 3 & 3 & -1 \end{vmatrix} + 3\begin{vmatrix} 3 & 2 & 3 \\ 4 & 2 & 4 \\ -1 & 3 & -1 \end{vmatrix}$$

$$= 1(-2-12) - 2(-3-4) + 3(9-2) + 2(0) + 3(0)$$

$$= -14 + 14 + 21 = 21$$